J. Fluid Mech. (1987), *vol.* 175, *pp.* 157–181
Printed in Great Britain

Nonlinear stability and statistical mechanics of flow over topography

By GEORGE F. CARNEVALE
Scripps Institution of Oceanography, University of California, San Diego, La Jolla, CA 92093, USA

AND JORGEN S. FREDERIKSEN
CSIRO, Division of Atmospheric Research, PB 1, Mordialloc, Victoria 3195, Australia

(Received 11 November 1985 and in revised form 23 June 1986)

The stability properties and stationary statistics of inviscid barotropic flow over topography are examined. Minimum enstrophy states have potential vorticity proportional to the streamfunction and are nonlinearly stable; correspondingly, canonical equilibrium based on energy and enstrophy conservation predicts mean potential vorticity is proportional to the mean streamfunction. It is demonstrated that in the limit of infinite resolution the canonical mean state is statistically sharp, that is, without any eddy energy on any scale, and is identical to the nonlinearly stable minimum enstrophy state. Special attention is given to the interaction between small scales and a dynamically evolving large-scale flow. On the β-plane, these stable flows have a westward large-scale component. Possibilities for a general relation between inviscid statistical equilibrium and nonlinear stability theory are examined.

1. Introduction

From the statistical study of the inviscid flow, there emerges an interesting correspondence between the canonical equilibrium theory and nonlinear stability theory as developed by Arnol'd (1965, 1969). To the extent that ergodic theory applies and long-term averages may be calculated from ensemble means, we should anticipate some such correspondence, simply because we would expect a system near a stable state to remain near it and reflect this in the time average. Our presentation develops both theories side by side, and attempts to show that the general stability theory implies the need for considering inviscid statistical theory in a more general form than has traditionally been the case in macroscopic fluid dynamics.

The specific model which we investigate here is barotropic quasi-geostrophic flow over topography on a β-plane for which there is a strong scale separation between the large-scale flow and the eddies. This model, which is a flat geometry approximation to flow over topography on a rotating sphere, has been of particular importance in the study of blocking (cf. Hart 1979). The large-scale motion is represented by a uniform zonal flow with the dynamical coupling to the eddies produced by form drag. It is found that the statistics of this model are very similar to the results for inviscid flow on a sphere where the dynamics of the zonal flow derive directly from the Euler equation in spherical geometry. The model can also be considered a representation of the local interaction between small-scale topographic features and large-scale flow in a large basin.

To a large degree many of the details of what is to follow are modifications of the statistics for flow over a fixed topography on an f-plane with no explicit 'large-scale' mean flow. So we begin by presenting the results for this simple model and then introduce the necessary modifications for the more complicated model with large scales.

The quasi-geostrophic equation for flow on the f-plane is simply the advection of potential vorticity

$$\frac{\partial q}{\partial t}+\nabla\cdot(\boldsymbol{v}q)=0, \tag{1.1}$$

by the divergenceless velocity field, $\boldsymbol{v}$, which can be written in terms of a stream-function, ψ, as

$$\boldsymbol{v}=(u,v)=\left(-\frac{\partial\psi}{\partial y},\frac{\partial\psi}{\partial x}\right).$$

Assuming that the variation in bottom topography, ΔH, is small relative to the total depth, D, the potential vorticity is given by

$$q=\nabla^2\psi+h, \tag{1.2}$$

where h is the spatial variation of the height of the bottom topography relative to the total depth in units of the Coriolis parameter f:

$$h(x,y)=f\frac{\Delta H}{D}.$$

It will also be convenient to represent the evolution equation as

$$\frac{\partial q}{\partial t}+J(\psi,q)=0, \tag{1.3}$$

where J, the Jacobian, is defined by

$$J(\psi,q)\equiv\frac{\partial\psi}{\partial x}\frac{\partial q}{\partial y}-\frac{\partial\psi}{\partial y}\frac{\partial q}{\partial x}.$$

For a derivation and discussion of the range of validity of this equation see, for example, Pedlosky (1979). Throughout what follows we assume periodic boundary conditions on ψ and h in both directions x and y.

For our purposes, the most important aspect of the evolution equation is its set of integral conservation laws. The conservation of potential vorticity on fluid particles expressed by (1.1) implies the conservation of all integrals

$$\iint F(q)\,\mathrm{d}x\,\mathrm{d}y,$$

were $F(q)$ is any function of q. That this is so with periodic boundary conditions is demonstrated directly from (1.1) and the fact that q here is also periodic. Of all these integrals, the most important is the quadratic invariant, the 'potential enstrophy':

$$Q=\frac{1}{2}\iint q^2\,\mathrm{d}x\,\mathrm{d}y. \tag{1.4a}$$

The only other invariant of the flow that is quadratic in ψ is the total energy,

$$E=\frac{1}{2}\iint (\nabla\psi)^2\,\mathrm{d}x\,\mathrm{d}y. \tag{1.4b}$$

Based on the existence of these quadratic invariants, nonlinear stability theory (cf. Arnol'd 1969) proves stability for the stationary state determined by

$$\mu\psi=q, \tag{1.5}$$

where certain restrictions are imposed on the proportionality constant μ. On the other hand, canonical statistical theory for a finite resolution version of the vorticity equation predicts that the ensemble average streamfunction satisfies (Salmon, Holloway & Hendershott 1976)

$$\mu\langle\psi\rangle = \langle q\rangle, \tag{1.6}$$

where the constant μ is in general different from the constant in the previous equation. The correspondence between these results is quite simply related to the fact that for this case both theories are based entirely on the conservation of energy and potential enstrophy (cf. Purini & Salusti 1984). In §4 we show how these 'states' become equivalent in the limit of infinite resolution (i.e. no ultraviolet spectral truncation).

In §5 we extend these results to flow on a β-plane with a uniform zonal flow. Energetics alone are sufficient to determine the local interaction of small (periodic) scales and the large-scale flow. The resulting equation for the large-scale zonal flow is the form drag equation. The coupled large-scale and small-scale equations conserve two invariants which are obtainable from the total energy and potential enstrophy via a two-scale analysis. Related investigations for unbounded flow on a sphere with finite resolution can be found in Frederiksen (1982) and Frederiksen & Carnevale (1986).

The comparison between nonlinear stability theory and canonical statistical mechanics raises some fundamental questions. The stability theory is far more general than the case of the solution (1.5) would suggest. The proof of the stability of that solution is based solely on energy and enstrophy conservation, but the full theory, which can treat a much wider class of solutions, is based on all possible conserved quantities of which there are an infinity for this dynamics (cf. Holm *et al.* 1985). Statistical equilibrium theory can also be extended in a similar way. Although there have been many arguments in favour of inviscid statistics based only on energy and enstrophy conservation, it would seem clear in the light of the following discussion that that cannot be defended in all cases. In §6 we investigate how the application of canonical equilibrium in a more general form may complete the correspondence with the general stability theory.

We begin our treatment with a review of the stability and statistical theories in §§2 and 3.

2. Nonlinear stability

Euler's equation can be written as

$$\frac{\partial q}{\partial t}+\hat{z}\cdot\nabla\psi\times\nabla q = 0, \tag{2.1}$$

in which form it becomes obvious that stationary solutions are such that the gradient of the streamfunction is always parallel to the gradient of q. Thus stationary solutions locally have the general form

$$\psi = F'(q), \tag{2.2}$$

where F can be an arbitrary function of q (cf. Fofonoff 1954) – the definition in terms of the derivative $F'(q) \equiv \mathrm{d}F/\mathrm{d}q$ is for later convenience. The simplest non-trivial such relationship is

$$\mu\psi = q, \tag{2.3}$$

that is

$$\mu\psi = \nabla^2\psi+h, \tag{2.4}$$

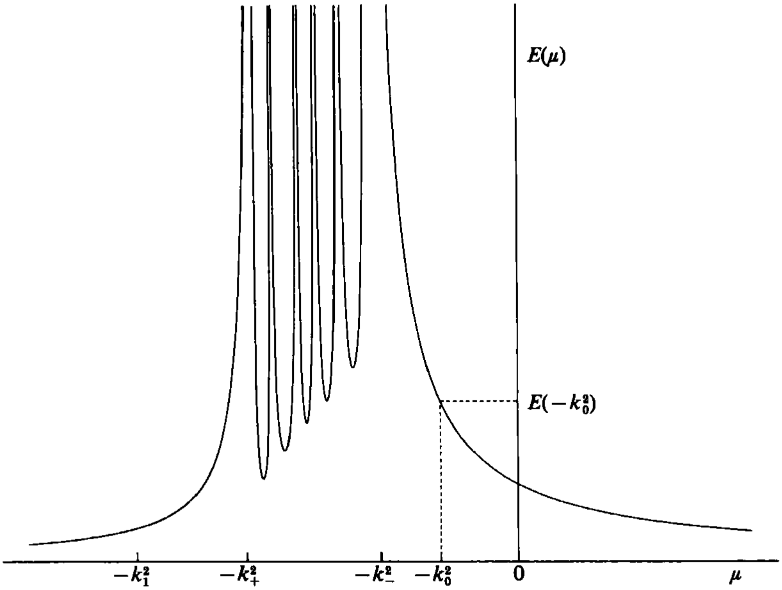

FIGURE 1. Energy in the stationary state $q = \mu\psi$. Schematic sketch of the relation between the total energy and the parameter μ as given by (2.6*a*). The topography is assumed to have non-zero spectral amplitudes only in the wavenumber range from k_- to k_+. A discrete spectral representation is assumed with resolved wavenumbers only in the range from k_0 to k_1. The solutions ψ^s are nonlinearly stable for all $\mu > -k_0^2$ (also for all $\mu < -k_1^2$ but these do not obtain in the physical limit $k_1 \to \infty$). There is an energy cutoff, $E(-k_0^2)$, above which there can be no physically relevant stable stationary state of this family.

with μ a constant. The solution to this equation, which we shall denote by ψ^s is readily obtained by Fourier transformation:

$$\psi_k^s = \frac{h_k}{\mu + k^2}. \tag{2.5}$$

This well-known solution has been investigated in some detail by Bretherton & Haidvogel (1976) among others. Here we shall expand on some particular details of this result that are relevant to the questions at hand.

All other quantities being equal, the energy and potential enstrophy of this solution can be viewed as a function of the parameter μ:

$$E^s(\mu) = \tfrac{1}{2}\sum_k \frac{k^2\,|h_k|^2}{(\mu+k^2)^2}, \tag{2.6a}$$

$$Q^s(\mu) = \tfrac{1}{2}\sum_k \frac{\mu^2\,|h_k|^2}{(\mu+k^2)^2}. \tag{2.6b}$$

For the following discussion we shall assume an artificial high-wavenumber truncation at $k_{max} = k_1$. Also there is a lower wavenumber cutoff k_0 defined by the size of the periodic domain. Thus only wavevectors such that $k_0^2 < k^2 < k_1^2$ are allowed, as would indeed be the case in a numerical simulation of the flow. The restriction to finite k_1 will be relaxed in §4.

In general the function $E^s(\mu)$ will have many discrete positive infinity points – one for each 'excited' wavenumber of the topography. For concreteness we begin the analysis by assuming the largest, k_+, and smallest, k_-, wavenumbers of the topography fall strictly between the allowed limits on k (i.e. $k_0 < k_- \leqslant k_+ < k_1$). Then the schematic in figure 1 represents the qualitative behaviour of $E^s(\mu)$.

Note that for a given energy E there are in general multiple stationary solutions $\mu(E)$. Not all the possible solutions need be stable. However, by using the method of Arnol'd (1969), one can demonstrate rather simply that there are two parameter regimes for which the flow is stable:

$$\mu > -k_0^2, \tag{2.7a}$$

and

$$\mu < -k_1^2. \tag{2.7b}$$

This method involves defining a norm in which an arbitrary perturbation of the stationary state remains bounded for all time. The result is referred to as nonlinear stability. In this case the norm is obtained simply by calculating the energy and enstrophy in the perturbed state. We write the perturbed state as

$$\psi = \psi^s + \delta\psi.$$

The norm is

$$\begin{aligned} Q - Q^s + \mu(E - E^s) &\equiv \frac{1}{2}\iint \mathrm{d}x\,\mathrm{d}y\{(\nabla^2\psi + h)^2 - (\nabla^2\psi^s + h)^2\} + \mu\,\frac{1}{2}\iint \mathrm{d}x\,\mathrm{d}y\{(\nabla\psi)^2 - (\nabla\psi^s)^2\} \\ &= \iint \mathrm{d}x\,\mathrm{d}y\{(\nabla^2\delta\psi)\,q^s + \mu\nabla\delta\psi\cdot\nabla\psi^s + \tfrac{1}{2}((\nabla^2\delta\psi)^2 + \mu(\nabla\delta\psi)^2)\} \\ &= \frac{1}{2}\iint \mathrm{d}x\,\mathrm{d}y\{(\nabla^2\delta\psi)^2 + \mu(\nabla\delta\psi)^2\}. \end{aligned} \tag{2.8}$$

This last line is obtained by using the definition (2.2) for ψ^s and an integration by parts. Since Q and E are constants of motion this last quantity is also conserved by the full nonlinear dynamics. In a Fourier representation this invariant is

$$Q - Q^s + \mu(E - E_s) = \tfrac{1}{2}\sum_k k^2(k^2 + \mu)\,|\delta\psi_k|^2. \tag{2.9}$$

Thus for μ in the range defined in (2.7) each term in the series is of the same definite sign, and hence we have defined a norm which is constant for all time. By this measure then, no matter how large the initial perturbation is it can never grow (or decay for that matter).

This nonlinear stability result implies nothing about the stability for the rest of the range of μ; that would have to be checked by other means such as linear stability calculation (for flow on a sphere Frederiksen & Carnevale (1985) find linear instability for the analogous range). Nevertheless, even if the stationary state turns out to be unstable for a particular value of μ the constancy of the quadratic quantity in (2.5) places strong restrictions on the manner in which the instability can develop as has been emphasized in the work of Vallis (1985) and Petroni, Pierini & Vulpiani (1986).

The analysis of Bretherton & Haidvogel (1976) is based on the fact that the branch of solutions with $\mu > -k_0^2$ represents solutions with minimal potential enstrophy Q for a given energy E. This is shown by taking the first and second functional variations of $Q + \mu E$, where μ plays the role of a Lagrange multiplier attached to the given value of E:

$$\delta(Q + \mu E) = -\sum_k k^2(q_k - \mu\psi_k)\,\delta\psi_k^*, \tag{2.10}$$

and

$$\delta^2(Q + \mu E) = \sum_k k^2(k^2 + \mu)\,|\delta\psi_k|^2. \tag{2.11}$$

The vanishing of the first variation actually defines the stationary solution (2.3), and the positivity of the second variation for $\mu > -k_0^2$ implies minimality. Also note that the branch with $\mu < -k_1^2$ represents a maximum enstrophy branch. Alternatively the solution in the stable range with positive (negative) μ has minimum (maximum) energy for a given Q as can be seen from the same variational equations. These points are also obvious directly from (2.9).

Although the branch with $\mu < -k_1^2$ is mathematically interesting, it does not survive in the physically interesting limit $k_1 \rightarrow \infty$. The only nonlinear stable solutions in this limit are on the minimum enstrophy branch. For our purposes it is worth pointing out at this point that for energy sufficiently large there may be no stable solution at all. For example if there is no topography of wavenumber k_0, then $E^s(-k_0^2)$ is finite and there is no nonlinear stable minimum enstrophy solution. A similar consideration holds for the maximum enstrophy branch. Nevertheless an ensemble average of the flow, even with E too large to be nonlinearly stable, will have a mean flow which is nonlinearly stable as discussed in the next section.

Finally, let us emphasize that since these flows are inviscid, a perturbation to the stationary solution, even if it is stable, can never decay. In fact, as noted above the constancy of the norm implies the perturbation is always bounded away from the unperturbed state. Furthermore, as will be demonstrated in the next section, the ensemble average and presumably the long time mean of the perturbation $\delta\psi$ is also non-vanishing – the fluctuations about the stable state have non-vanishing expectation (for finite resolution).

3. Inviscid statistical equilibrium

The inviscid statistical mechanics of quasi-geostrophic flow over topography is developed in Salmon *et al.* (1976). A prerequisite for such a statistical treatment is that the condition for Liouville's theorem be satisfied, that is, the motion in the phase space of the chosen variables must be incompressible. For our problem it is convenient to use the Fourier components of the streamfunction $\psi_{\boldsymbol{k}}$ or equivalently the relative vorticity $\zeta_{\boldsymbol{k}} = -k^2\psi_{\boldsymbol{k}}$. The equation of motion for these variables is given by

$$\dot{\zeta}_{\boldsymbol{k}} = \sum_{\boldsymbol{p},\boldsymbol{q}} A_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}}\, \zeta_{\boldsymbol{p}}(\zeta_{\boldsymbol{q}} + h_{\boldsymbol{q}}), \tag{3.1}$$

where the interaction coefficient is

$$A_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}} = (p_x q_y - p_y q_x)\, p^{-2}\, \delta_{\boldsymbol{k},\boldsymbol{p}+\boldsymbol{q}}. \tag{3.2}$$

Note that

$$A_{\boldsymbol{k},\boldsymbol{k},\boldsymbol{q}} = A_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{k}} = 0, \tag{3.3}$$

for all $\boldsymbol{p}$ and $\boldsymbol{q}$, or in other words there is no self interaction among Fourier modes; this is sufficient for incompressibility in phase space, that is

$$\sum_{\boldsymbol{k}\in K} \frac{\partial \dot{\psi}'_{\boldsymbol{k}}}{\partial \psi'_{\boldsymbol{k}}} + \frac{\partial \dot{\psi}''_{\boldsymbol{k}}}{\partial \psi''_{\boldsymbol{k}}} = 0. \tag{3.4}$$

Here the set K is a reduced set of wavenumbers which takes into account the correct enumeration of independent degrees of freedom. The point is that we must avoid the redundancy implied in the Hermiticity constraint

$$\psi_{\boldsymbol{k}} = \psi^*_{-\boldsymbol{k}}, \tag{3.5}$$

which follows from the reality of $\psi(x, y)$. This may be accomplished by taking K to be the set of wavevectors such that for each $\boldsymbol{k}$ which appears in the list of the corresponding $-\boldsymbol{k}$ does not appear. Here the real and imaginary parts of the streamfunction amplitude are denoted by prime and double prime, and these are the actual phase space variables that we use throughout. When we write the probability distribution in this phase space, $P(\{\psi_{\boldsymbol{k}}\})$, the independent variables are the real and imaginary parts of ψ with $\boldsymbol{k}$ restricted to the set K. However, all the sums in what follows are unrestricted and include all wavevectors; the identification (3.5) remains implicitly understood.

It is interesting to note that the ψ's are not canonical variables even though the system is Hamiltonian in the sense that a transformation to canonical variables is possible. The important thing is that the ψ's form an incompressible phase space so the methods of statistical mechanics can be applied directly in these variables. This will permit exact closed-form analytical results, whereas the nonlinear transformation to canonical coordinates would make that impossible (cf. Salmon 1982; Henyey 1983).

The canonical equilibrium for quasi-geostrophic flow is traditionally based on the conservation of energy and potential enstrophy and can be expressed as

$$P(\{\psi_{\boldsymbol{k}}\}) \propto \mathrm{e}^{-aE-bQ}. \tag{3.6}$$

For a general review of the theory of the application of equilibrium statistical mechanics to two-dimensional fluids see Kraichnan & Montgomery (1980) (also recommended are Basdevant & Sadourny 1975, and Orzsag 1970). The use of the energy and enstrophy invariants alone usually raises many questions since the vorticity equation that we started from, (1.1), conserves the integral of any function of the vorticity, not only potential enstrophy. An immediate response here is that (3.6) is the proper choice for the truncated dynamics because the finite resolution model does not conserve the higher-order vorticity integrals. Strictly speaking this is correct; however, it is not completely satisfying and we shall address this question further in §§6 and 7, where we explore the consequences of generalizing (3.6) to include other invariants – in this model with periodic boundary conditions the conservation of total vorticity is a trivial modification and will not be explicitly included here.

Because the distribution (3.6) is Gaussian its moments are readily calculated. The mean and variance are given by (cf. Salmon *et al.* 1976)

$$\langle \psi_{\boldsymbol{k}} \rangle = \frac{h_{\boldsymbol{k}}}{(a/b)+k^2}, \tag{3.7a}$$

$$\langle (\zeta_{\boldsymbol{k}} - \langle \zeta_{\boldsymbol{k}} \rangle)(\zeta_{\boldsymbol{p}} - \langle \zeta_{\boldsymbol{p}} \rangle) \rangle = \frac{k^2}{a+bk^2}\,\delta_{\boldsymbol{k},\,-\boldsymbol{p}}. \tag{3.7b}$$

In the present context, the first thing to note is that the functional form of $\langle \psi \rangle$ is the same as ψ^{s} as defined in the previous section with μ replaced by a/b.

Interestingly, even though this ensemble represents a maximum entropy state (cf. Carnevale, Frisch & Salmon 1981; Montgomery 1976, 1985), the statistics are in general inhomogeneous and anisotropic and the average ψ is non-trivial with as much structure as in the topography. However, all the anisotropy and inhomogeneity is contained in the mean field; that is, the fluctuating eddy energy given by (3.3) is isotropic and homogeneous independent of the mean field or underlying topography.

The parameters a and b are determined by the prescribed values of E and Q for the mean energy and potential enstrophy of the ensemble. The defining relations are implicit and take the form:

$$\left.\begin{aligned} E &= \tfrac{1}{2}\sum_k k^2\langle|\psi_k|^2\rangle \\ &= \tfrac{1}{2}\sum_k k^2(\langle|\psi_k-\langle\psi_k\rangle|^2\rangle+|\langle\psi_k\rangle|^2) \\ &= \tfrac{1}{2}\sum_k \frac{1}{a+bk^2}+\frac{k^2|h_k|^2}{((a/b)+k^2)^2}. \end{aligned}\right\} \tag{3.8a}$$

$$\left.\begin{aligned} Q &= \tfrac{1}{2}\sum_k \langle|\zeta_k+h_k|^2\rangle \\ &= \tfrac{1}{2}\sum_k \langle|\zeta_k-\langle\zeta_k\rangle|^2\rangle+|\langle\zeta_k\rangle+h_k|^2 \\ &= \tfrac{1}{2}\sum_k \frac{k^2}{a+bk^2}+\frac{(a/b)^2|h_k|^2}{((a/b)+k^2)^2}. \end{aligned}\right\} \tag{3.8b}$$

For emphasis, the decomposition into mean and eddy terms is explicitly displayed in these expressions. Katz (1967) demonstrates that for all physically realizable E and Q these implicit equations have a unique solution $a(E,Q)$ and $b(E,Q)$. From (3.7b) it is clear that physical realizability implies that

$$a+bk^2 > 0, \tag{3.9}$$

for all allowed values of k. Thus the canonical equilibrium is such that with $\mu^{eq} \equiv a/b$ we have

$$\left.\begin{aligned} \mu^{eq} &> -k_0^2 \quad \text{if } b>0, \\ \mu^{eq} &< -k_1^2 \quad \text{if } b<0. \end{aligned}\right\} \tag{3.10}$$

Therefore the values of μ^{eq} are restricted to precisely the same range as defined by the stable branches of μ^s derived in the previous section. However, for a given value of E, $\mu^{eq}(E,Q)$ is not the same as $\mu^s(E)$ except when Q is an extremum (i.e. for $Q=Q^s$).

Comparison of E^s given by (2.6) to E^{eq} given by (3.8) shows that for $\mu > -k_0^2$ (the minimum enstrophy branch) we have for equal energies, $E^s = E^{eq}$, that $\mu^s \leqslant \mu^{eq}$ and of course $Q^{eq} \geqslant Q^s = Q_{min}$. Similarly for $\mu \leqslant -k_1^2$ we have $\mu^{eq} \leqslant \mu^s$ and $Q \leqslant Q^s = Q_{max}$. Thus an energy preserving perturbation to an equilibrium state has non-vanishing mean streamfunction. That is, since Q is an extremum in the stable state we have for the perturbed state $Q \neq Q^s$ and $\mu^{eq}(E,Q) \neq \mu^s(E)$. If ensemble averages can be replaced by time averages we then have that the long time average of the perturbed state differs from the stable state which was perturbed. Indeed the long time average of the perturbation cannot vanish. This is the case for the finite resolution model. In contrast, as we shall show in the next section, in the limit of infinite resolution the minimum enstrophy state and the canonical equilibrium are identical.

4. Infinite resolution

In this section we show that the statistical mechanical equilibrium state 'collapses' onto a corresponding Arnol'd stable state in the limit of infinite resolution. That is, in the limit $k_1 \to \infty$ the eddy energy vanishes at wavenumbers k, such that $k_0 < k < \infty$ leaving the sharply defined minimum enstrophy state.

For the purpose of the following calculations we shall assume the wavenumber band from k_0 to k_1 is continuous so that we can convert the sums to integrals and obtain convenient closed form expressions for the eddy energy terms. The particulars of the calculation for the discrete case add little to the overall picture and will not be included here. Thus evaluating the expressions in (3.8), we have

$$E = \frac{\pi}{2b} \ln \frac{k_1^2+\mu}{k_0^2+\mu} + \frac{1}{2} \iint d^2k \, \frac{k^2 |h_k|^2}{(\mu+k^2)^2}, \tag{4.1}$$

$$Q = \frac{\pi}{2b} \left[(k_1^2-k_0^2) - \mu \ln \frac{k_1^2+\mu}{k_0^2+\mu} \right] + \frac{1}{2} \iint d^2k \, \frac{\mu^2 |h_k|^2}{(\mu+k^2)^2}, \tag{4.2}$$

where $\mu = a/b = \mu^{\mathrm{eq}}(E,Q)$. Without the topographic terms these results are given in Fox & Orszag (1973) and Kraichnan (1975).

We first consider the case where the is no topography on the smallest or largest allowed scales. The limit of infinite resolution in the absence of topography is discussed by Kraichnan (1975) and Basdevant & Sadourny (1975); here we show the modifications due to the presence of topography. If the parameters a and b both remain finite as $k_1 \to \infty$ then the eddy energy would diverge which is inconsistent for a prescribed finite E. Thus either a or b or both must diverge in this limit. We find that (4.1) and (4.2) can be simultaneously satisfied with the ratio μ remaining finite in this limit. The leading asymptotic terms are calculated by assuming a finite μ and then calculating b. As may have been anticipated from our previous discussion, the form of the asymptotic result depends on the size of E relative to $E^{\mathrm{s}}(-k_0^2)$, which is the maximum possible energy of a minimum enstrophy state. The solution is broken into two cases:

Case I. $E \leqslant E^{\mathrm{s}}(-k_0^2)$

$$\mu^{\mathrm{eq}}(E,Q) \to \mu^{\mathrm{s}}(E), \tag{4.3a}$$

$$b \to \frac{\pi k_1^2}{2(Q-Q^{\mathrm{s}}(\mu^{\mathrm{s}}))}, \tag{4.3b}$$

Case II. $E \geqslant E^{\mathrm{s}}(-k_0^2)$

$$\mu^{\mathrm{eq}}(E,Q) \to -k_0^2 + k_1^2 \exp\{-2b(E-E^{\mathrm{s}}(-k_0^2))/\pi\}, \tag{4.4a}$$

$$b \to \frac{\pi k_1^2}{2} [Q - Q^{\mathrm{s}}(-k_0^2) - k_0^2(E-E^{\mathrm{s}}(-k_0^2))]^{-1}. \tag{4.4b}$$

If there were no topography this would reduce to the pure 'turbulence' problem, which is represented by case II with $E^{\mathrm{s}} = 0$ and $Q^{\mathrm{s}} = 0$. That case is examined in Kraichnan (1975). Without topography all of the energy is to be found at $k = k_0$, while the enstrophy is split with a portion $k_0^2 E$ at k_0 and the rest at $k = \infty$.

With topography, case I is such that $\langle \psi \rangle$ becomes identical with ψ^{s} and there is no eddy energy at any scale. The mean state $\langle \psi \rangle$ then has enstrophy Q^{s} which must be less than or equal to Q, since this is a minimum enstrophy state. Any amount of Q exceeding Q^{s} appears at $k = \infty$. To demonstrate this we compute the enstrophy contained in the spectrum from κ to k_1, where κ is such that $k_+ < \kappa < k_1 < \infty$ (recall k_+ is the highest wavenumber of the topography). Then we take the limit $k_1 \to \infty$ followed by the limit $\kappa \to \infty$. Combining (4.2) and (4.3) for case I produces

$$\lim_{\kappa\to\infty} \left[\lim_{k_1\to\infty} \frac{\pi}{2b} \left[(k_1^2-\kappa^2) - \mu \ln \frac{k_1^2+\mu}{\kappa^2+\mu} \right] \right] = \lim_{\kappa\to\infty} (Q-Q^{\mathrm{s}}) = Q-Q^{\mathrm{s}}. \tag{4.5}$$

For case II, the energy E is larger than can be accommodated by a minimum enstrophy state. The canonical equilibrium establishes a mean flow with energy equal to that maximum allowed for a stable state. The excess energy, $E-E^{s}(-k_0^2)$, is deposited at the smallest wavenumber, k_0. The excess enstrophy not contained in the mean field can be found in two places: at k_0 there is a net enstrophy of $k_0^2(E-E^{s})$, and at $k=\infty$ the remaining $Q-Q^{s}-k_0^2(E-E^{s})$ is to be found. These assertions follow from (4.1), (4.2) and (4.4) as above:

the eddy energy at k_0 is

$$\lim_{\kappa\to k_0}\left(\lim_{k_1\to\infty}\int_{k_0}^{\kappa} \mathrm{d}k\,\frac{\pi k}{a+bk^2}\right)=\lim_{\kappa\to k_0}\left(\lim_{k_1\to\infty}\frac{\pi}{2b}\ln\frac{\kappa^2+\mu}{k_0^2+\mu}\right)$$

$$=\lim_{\kappa\to k_0}(E-E^{s}(-k_0^2))$$

$$=E-E^{s}(-k_0^2); \tag{4.6}$$

the eddy enstrophy at k_0 is

$$\lim_{\kappa\to k_0}\left(\lim_{k_1\to\infty}\int_{k_0}^{\kappa} \mathrm{d}k\,\frac{\pi k^2}{a+bk^2}\right)=\lim_{\kappa\to k_0}\left[\lim_{k_1\to\infty}\frac{\pi}{2b}\left[(\kappa^2-k_0^2)-\mu\ln\frac{\kappa^2+\mu}{k_0^2+\mu}\right]\right]$$

$$=k_0^2(E-E^{s}(-k_0^2)); \tag{4.7}$$

and the eddy enstrophy at $k=\infty$ is

$$\lim_{\kappa\to\infty}\left[\lim_{k_1\to\infty}\int_{\kappa}^{k_1} \mathrm{d}k\,\frac{\pi k^2}{a+bk^2}\right]=\lim_{\kappa\to\infty}\left[\lim_{k_1\to\infty}\frac{\pi}{2b}\left[(k_1^2-\kappa^2)-\mu\ln\frac{k_1^2+\mu}{\kappa^2+\mu}\right]\right]$$

$$=Q-Q^{s}-k_0^2(E-E^{s}(-k_0^2)). \tag{4.8}$$

The restrictions on k_- and k_+ can be relaxed without difficulty. Setting $k_+=k_1$ causes no change in either case I or II (assuming the topography falls of sufficiently rapidly at $k=\infty$. Setting $k_-=k_0$ yields $E(-k_0^2)=\infty$, and so case II no longer applies, but case I is unaltered.

5. Flow on a β-plane

In most geophysical fluid contexts the problem of flow over topography also involves the effects of differential rotation rate and interaction with the large-scale flows that drive the system. For our purposes the β-plane approximation will suffice to model the effects of differential rotation on synoptic scale flow features. It is interesting to consider how best to model the interaction of these 'small' scale features with large or basin scale features. Here we shall approach the problem from a view point which emphasizes the role of the integral invariants, although one can also derive the same result from a two-scale analysis (cf. Hart 1979, for completeness we provide such a derivation in Appendix A).

We begin by considering periodic flow on a β-plane. The evolution equation is then

$$\frac{\partial\zeta}{\partial t}+J(\psi,\zeta+h+\beta y)=0. \tag{5.1}$$

With periodic conditions on h and ψ, this equation does not have a potential enstrophy invariant. The rate of change of the potential enstrophy in the 'small scales' is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\iint (\zeta+h)^2\,\mathrm{d}x\,\mathrm{d}y = -\beta\iint h\,\frac{\partial\psi}{\partial x}\,\mathrm{d}x\,\mathrm{d}y, \tag{5.2}$$

and the rate of change of the full potential enstrophy including the large-scale variation βy is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\iint (\zeta+h+\beta y)^2\,\mathrm{d}x\,\mathrm{d}y = -\beta l\int v(\zeta+h)|_{y=l}\,\mathrm{d}x, \tag{5.3}$$

where the scale of the periodic domain, l, is hereafter taken to be unity. As an aside, note that if $h\equiv 0$ then the relative enstrophy (5.2) is conserved on the periodic β-plane; this implies that the canonical statistics of the periodic β-plane are independent of β.

The loss of the enstrophy invariant when topography is present can be circumvented by explicitly including large-scale flow. A self-consistent model can be achieved by incorporating a large-scale east–west flow U. The full streamfunction is now $\Psi\equiv\psi-Uy$. The resulting evolution equaton for the 'small scales' then is

$$\frac{\partial\zeta}{\partial t}+J(\psi-Uy,\zeta+h+\beta y) = 0. \tag{5.4}$$

Of course, we still need an evolution equation for U. If U is simply held fixed then the energy is no longer conserved. The rate of change of energy in the 'small scales' is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\iint (\nabla\psi)^2\,\mathrm{d}x\,\mathrm{d}y = -U\iint h\,\frac{\partial\psi}{\partial x}\,\mathrm{d}x\,\mathrm{d}y. \tag{5.5}$$

The potential enstrophy rate equations (5.2) and (5.3) are unaffected by the addition of U; thus with constant U (5.4) conserves neither energy nor potential enstrophy. Actually there is still a quadratic invariant for this system, but it is a combination of energy and enstrophy; that conserved quantity is

$$\iint \mathrm{d}x\,\mathrm{d}y\left[(\nabla\psi)^2-\frac{U}{\beta}(\zeta+h)^2\right], \tag{5.6}$$

as can be seen by comparing (5.2) with (5.5) (Vallis 1985).

From (5.5) we immediately see that the conservation of the full energy,

$$E_\Psi = \tfrac{1}{2}U^2+\frac{1}{2}\iint (\nabla\psi)^2\,\mathrm{d}x\,\mathrm{d}y, \tag{5.7}$$

is achieved if U evolves according to

$$\dot U = \iint h\,\frac{\partial\psi}{\partial x}\,\mathrm{d}x\,\mathrm{d}y. \tag{5.8}$$

This also recovers the conservation of potential enstrophy defined by

$$Q_\Psi = \beta U+\frac{1}{2}\iint (\zeta+h)^2\,\mathrm{d}x\,\mathrm{d}y. \tag{5.9}$$

It is shown in Appendix A that these forms for E and Q also result from a two-scale analysis. A large-scale northward flow, V, may also be included in the model without difficulty. The conserved enstrophy, Q_Ψ, would be unaltered since there is a no large-scale variation of the Coriolis parameter to couple to V, while the energy, E_Ψ, would include additionally $\frac{1}{2}V^2$, The conservation laws then provide an evolution equation for V analogous to (5.8). Carrying this modification through the following discussion would add little and is neglected for simplicity.

Thus (5.4) and (5.8) provide a closed system describing the interaction between large and small scales and maintaining the important quadratic invariants of the flow. These are in essence the same equations used by Hart (1979) and Charney & DeVore (1979) in the study of multiple equilibria flows. U plays the role of the Y_0^l spherical harmonics for flow on a sphere, and there is a close correspondence between the statistics of the inviscid zonal flow on the sphere and the statistics of U (cf. Frederiksen 1982; Frederiksen & Carnevale 1986).

Building on our previous results for the f-plane we can quickly develop the nonlinear stability and canonical equilibrium of this model. According to (5.4) the stationary solution must be such that the gradient of the full streamfunction, $\Psi \equiv \psi - Uy$, is everywhere parallel to the gradient of $q \equiv \zeta + h + \beta y$. The simplest possible non-trivial solution is again the linear proportionality, $\mu\Psi = \zeta = h + \beta y$, or more explicitly

$$\mu(\psi^s - U^s y) = \nabla^2\psi^s + h + \beta y. \tag{5.10}$$

Equating the periodic and large-scale pieces separately we have

$$\mu = -\frac{\beta}{U^s}, \tag{5.11}$$

and

$$\mu\psi^s = \nabla^2\psi^s + h.$$

Substituting ψ^s in (5.8) yields the consistency check $\dot{U} = 0$ – this is carried out most simply in the Fourier representation. We can again view the parameter μ as a function of E or vice versa. The definitions of E^s and Q^s are now modified to

$$E^s(\mu) = \frac{1}{2}\frac{\beta^2}{\mu^2} + \tfrac{1}{2}\sum_k \frac{k^2\,|h_k|^2}{(\mu+k^2)^2}, \tag{5.12a}$$

and

$$Q^s(\mu) = -\frac{\beta^2}{\mu} + \tfrac{1}{2}\sum_k \frac{\mu^2\,|h_k|^2}{(\mu+k^2)^2}. \tag{5.12b}$$

These formulae may be interpreted as modifications in which a $k = 0$ component has been added to the topography. Thus, since we now have topography at the largest allowed scale there is always an allowed minimum enstrophy solution for any given E; this is made clear in the modified schematic in figure 2, where k_0, k_-, k_+, and k_1 retain there previous definitions referring only to the periodic functions ψ and h. There is a 'spectral gap' between the large-scale, $k = 0$, and the scales with $k \geqslant k_0$.

Nonlinear stability is established by adding the large-scale terms to the computation of the perturbation energy and enstrophy. The result is

$$Q_\Psi - Q^s + \mu(E_\Psi - E^s) \equiv \tfrac{1}{2}\sum_k k^2(k^2+\mu)\,|\delta\psi_k|^2 + \tfrac{1}{2}\mu(\delta U)^2, \tag{5.13}$$

where the relation (5.11) has been used to obtain this compact form. This is positive definite if $\mu > 0$ and negative definite if $\mu < -k_1^2$. Thus the stability range is decreased by shifting the largest scale from k_0 to $k = 0$. Note that the positive μ branch corresponds to negative U^s (i.e. westward flow). In the high resolution limit, $k_1 \to \infty$ the stable eastward flow branch no longer exists.

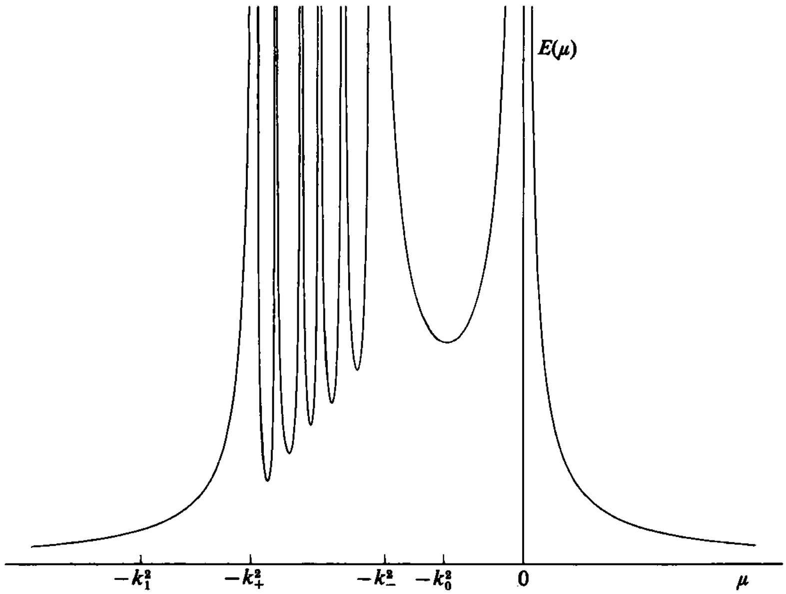

FIGURE 2. Energy in the stationary state $q = \mu\Psi$. As in figure 1 except here a large-scale flow (U) and 'topography' (βy) are included. The range of the physically relevant nonlinearly stable solutions is reduced to $\mu > 0$. There is no energy cutoff, that is, for any given energy there is a corresponding nonlinearly stable state.

We can compare these stability results with the canonical statistical equilibrium. The probability distribution is now defined by

$$P(\{\psi_k\}, U) \propto \exp\{-aE_\Psi - bQ_\Psi\}$$
$$\propto \exp\left\{\tfrac{1}{2}a\left(U+\frac{b}{a}\beta\right)^2\right\} \exp\{-aE_\psi - bQ_\psi\}. \tag{5.14}$$

Since this distribution is Gaussian the calculation of the averages is again straightforward. The average and variance of ψ are given by the formulae previously displayed, (3.7). The statistics of the large-scale flow are characterized completely by

$$\langle U\rangle = -\frac{b}{a}\beta = -\frac{\beta}{\mu^{\mathrm{eq}}}, \tag{5.15}$$

where again $\mu^{\mathrm{eq}} = a/b$, and

$$\langle (U-\langle U\rangle)^2\rangle = \frac{1}{a}. \tag{5.16}$$

Paradoxically, these results for the statistics of U do not explicitly involve the topography h. In the limit $h \to 0$, (5.15) unambiguously predicts a mean U; however, for h identically zero, U is an invariant with arbitrary value. For the ideal case with h identically zero, the proper distribution allows U to be independently specified unlike (5.14). The point is that the presence of h, no matter how small, breaks the east–west Galilean invariance and U will equilibrate in accordance with (5.16) and (5.17). However, the smaller the value of h, the longer it will take for the equilibration process to approximate the $t = \infty$ statistics.

Note that (5.16) implies that $a > 0$ and similarly the positivity of the eddy energy implies $a + bk^2 > 0$ for all k from k_0 to k_1. Combining these restrictions we see that the average solutions have either $\mu^{\mathrm{eq}} > 0$ (westward flow of arbitrary amplitude) or

$\mu^{eq} < -k_1^2$ (eastward flow of small amplitude). This parameter range for equilibrium is again the same as for nonlinear stability.

The infinite resolution limit is also only a slight modification of the previous result. Since U acts as a $k = 0$ flow and βy is essentially a $k = 0$ topography component, there is always a minimum enstrophy solution as pointed out above. Thus the asymptotic limit is given by case I of the previous section. For a given total energy $E = E_\psi$ and potential enstrophy $Q = Q_\psi$ we have:

$$\left.\begin{aligned} &\mu^{eq}(E,Q) \to \mu^s(E), \\ &b \to \frac{\pi k_1^2}{2(Q - Q^s(\mu^s))}. \end{aligned}\right. \tag{5.17}$$

Thus in this limit all of the energy is contained in the mean flow, with the residual potential enstrophy again found at $k = \infty$ and no eddy energy at any finite scale:

$$\left.\begin{aligned} \langle U\rangle^{eq} &\to U^s, \\ \langle (U + \langle U\rangle)^2\rangle &\to 0, \\ \langle \psi_k\rangle &\to \frac{h_k}{\mu^s + k^2}, \\ \langle |(\zeta_k - \langle\zeta_k\rangle)|^2\rangle &\to 0. \end{aligned}\right\} \tag{5.18}$$

Thus at all finite scales the canonical equilibrium is statistically sharp and equivalent to the nonlinearly stable state of the same energy.

6. A general relationship between stability and canonical equilibrium

In the case of the family of stationary solutions given by $\psi = (1/\mu)q$, we have demonstrated a relation between nonlinear stability theory and canonical equilibrium which becomes an equivalence in the limit of infinite resolution. This one example leads us to investigate the possiblity of a more general relationship. For this discussion we put aside the details of the definition of q and the particulars of the boundary conditions on ψ. The general stationary state is a solution of

$$\psi^s = F'(q^s). \tag{6.1}$$

In the case with $F = \frac{1}{2}q^2/\mu$, the stability is ensured by the conservation of energy and enstrophy. For the stability of the general steady state we examine energy and the general vorticity integral invariant defined by $Q_F = \iint F(q)$. As emphasized by Bretherton & Haidvogel (1976), the general stationary state (6.1) is an extremal point of Q_F for a given energy E. As in (2.8), we may try to find a norm which is preserved for all time for arbitrary initial perturbation. Following the analogous calculation in §2, we now have

$$\left.\begin{aligned} &E - E^s + Q_F - Q_F^s \\ &\equiv \iint \mathrm{d}x\,\mathrm{d}y\, \nabla\delta\psi\cdot\nabla\psi^s + \tfrac{1}{2}(\nabla\delta\psi)^2 + F'(q^s)\,\delta q + \tfrac{1}{2}F''(q^s)\,\delta q^2 + O(\delta q^3) \\ &= \iint \mathrm{d}x\,\mathrm{d}y\,(-\psi^s + F'(q^s))\,\delta q + \tfrac{1}{2}(\nabla\delta\psi)^2 + \tfrac{1}{2}F''(q^s)\,\delta q^2 + O(\delta q^3) \\ &= \frac{1}{2}\iint \mathrm{d}x\,\mathrm{d}y\,\delta q(-\nabla^{-2} + F''(q^s))\,\delta q + O(\delta q^3). \end{aligned}\right\} \tag{6.2}$$

The integral quadratic in δq is actually conserved by the linearized dynamics (see Appendix B for details), and hence we immediately have linear stability if this quadratic piece is of definite sign, that is, if the operator

$$-\nabla^{-2}+F''(q^{s})$$

is of definite sign. Note that the integral quadratic in δq is not in general conserved by the full nonlinear dynamics; nevertheless, useful sufficient conditions for nonlinear stability can still be obtained as demonstrated by Arnol'd (1965, 1969). One result is that positive definiteness of this quadratic integral is in fact also sufficient for nonlinear stability. It follows that positivity of $F''(q^{s})$ for all $q^{s}(x,y)$ is also sufficient for nonlinear stability (Arnol'd 1969, theorem 1). If $F''(q^{s})$ is positive for all space, the integral quadratic in δq is bounded by the 'size' (as defined below) of the initial perturbation for all time. We shall concentrate here on this result which is perhaps the simplest to deal with in terms of its implications for the connection of inviscid statistical equilibrium and stability.

We begin by reviewing Arnol'd's (1969) argument. Assume that F is such that

$$0<c\leqslant F''(q^{s})\leqslant C<\infty. \tag{6.3}$$

Arnol'd uses the richness of the infinity of dynamical invariants in two-dimensional flow to create a measure of the deviation from the stationary state which is bounded for all time no matter how large the initial perturbation. First, note that for a given stationary state q^{s} there is a freedom in the choice of the function F that is used in the defining relation (6.1). In fact, F may be replaced by any function, say G, which agrees with F over the range of values, $q^{s}_{\min}\leqslant q^{s}\leqslant q^{s}_{\max}$, taken on by the solution q^{s}. Thus ψ^{s} may be equivalently defined as the appropriate solution to

$$\psi=G'(q). \tag{6.1$'$}$$

Futhermore, we can choose a $G(q)$ which is the same as F over the range of q^{s} and which also satisfies the inequality (6.3) for *all* q. That is, we can choose

$$0<c\leqslant G''(q)\leqslant C<\infty, \tag{6.4}$$

for all q. In fact, we could further restrict G to be smooth through its second derivative. For example, a specific choice might be

$$G(q)=\begin{cases} F(q^{s}_{\min})+F'(q^{s}_{\min})\,(q-q^{s}_{\min})+\tfrac{1}{2}F''(q^{s}_{\min})\,(q-q^{s}_{\min})^{2} & (q<q^{s}_{\min}),\\ F(q) \quad (q^{s}_{\min}\leqslant q\leqslant q^{s}_{\max}), & \\ F(q^{s}_{\max})+F'(q^{s}_{\max})\,(q-q^{s}_{\max})+\tfrac{1}{2}F''(q^{s}_{\max})\,(q-q^{s}_{\max})^{2} & (q^{s}_{\max}<q).\end{cases} \tag{6.5}$$

This specific construction is given for clarity; all we will use below is the smoothness through the second derivative. If we replace Q_F by $Q_G\equiv\iint G(q)$ in (6.2), we can then make an estimate of the resulting terms of $O(\delta q^{3})$. Integrating over q twice in (6.4), we obtain

$$\tfrac{1}{2}c\,\delta q^{2}\leqslant G(q)-G(q^{s})-G'(q^{s})\leqslant \tfrac{1}{2}C\,\delta q^{2}. \tag{6.6}$$

For convenience define

$$H=\iint \mathrm{d}x\,\mathrm{d}y\,\tfrac{1}{2}(\nabla\psi)^{2}+G(q). \tag{6.7}$$

From (6.6) it then follows that

$$\frac{1}{2}\iint(\nabla\delta\psi)^{2}+c\,\delta q^{2}\leqslant H-H^{s}\leqslant\frac{1}{2}\iint(\nabla\delta\psi)^{2}+C\delta q^{2}, \tag{6.8}$$

for any perturbation $\delta\psi$ of ψ^s. Nonlinear stability then follows by the dynamical invariance of H (see Arnol'd 1969 for details). That is, Q_G is invariant and thus so is H; then since each inequality in (6.8) holds for all time we can evaluate the right-hand inequality at the initial time, $t = 0$, and this will bound the left-hand side evaluated at any time t. This yields

$$\frac{1}{2}\iint (\nabla\delta\psi(t))^2 + c(\delta q(t))^2 \leqslant H(t) - H^s = H(0) - H^s \leqslant \frac{1}{2}\iint (\nabla\delta\psi(0))^2 + C(\delta q(0))^2. \quad (6.9)$$

Thus the quantity on the left is a norm which is bounded for all time by the initial deviation of the initial conditions as measured by the quantity on the right. This holds for all perturbations no matter how large and is thus considerably stronger than linear stability.

Note that although the choice of G is not unique, the solution to (6.1′) is unique. Since G'' satisfies (6.4) for all q any solution to (6.1′) satisfies (6.8), which then proves that once the choice of G is made (6.1′) has only one solution, q^s. This can be seen replacing ψ in (6.8) by the alternate solution and then interchanging the roles of the two solutions. The sign of ΔH would change under this interchange, but the bounds would be the same positive numbers, thus providing a contradiction.

We now turn to the statistical theory. Canonical equilibrium theory based on energy and enstrophy conservation has only one possible mean state:

$$\langle\psi\rangle = \frac{1}{\mu}\langle q\rangle. \quad (6.10)$$

Since there are actually an infinity of conserved vorticity integrals, choosing only the enstrophy to define the canonical statistics has always been a troublesome point. Nonlinear stability theory presents us with an inconsistency in this statistical theory. To make this argument as forcefully as possible, consider the following thought experiment. For sake of argument, assume F' is nonlinear and the stability criterion (6.3) is satisfied. Apply a small perturbation to the stationary state (6.1), and perform a long-term average of the evolution. It would be inconsistent to predict that the average would produce the result (6.10) when Arnol'd's theorem guarantees that ψ can never be far from (6.1) especially considering very small initial perturbations.

These considerations indicate that restricting canonical equilibrium to be based only on energy and enstrophy conservation is not valid for all situations. In fact, general equilibrium statistical theory suggests it is important to consider all possible invariants. We may consider a generalization of the canonical equilibrium in which invariants other than energy and enstrophy are used. Here we shall demonstrate that it is possible to construct more general canonical distributions which are consistent with the existence of nonlinearly stable states other than the minimum enstrophy state. With H and G defined as above, we consider the stationary distribution

$$P = N\,\mathrm{e}^{-aH}. \quad (6.11)$$

Here for the sake of argument we are absorbing the multiplier b in the definition of G. The inequality (6.8) assures that the distribution is normalizable, and N is fixed by unit normalization. Furthermore, normalizability requires that a be positive. Equation (6.8) immediately leads to

$$N\exp\left\{-a\left(H^s + C\iint\delta q^2\right)\right\} \leqslant P \leqslant N\exp\left\{-a\left(H^s + c\iint\delta q^2\right)\right\}. \quad (6.12)$$

Thus P achieves its global maximum when δq vanishes everywhere – the state ψ^s is the unique state of maximum likelihood of the distribution (6.11). Thus there is a direct correspondence between the stationary state satisfying Arnold's sufficient stability criterion (6.3) and the state of maximum likelihood of this distribution (6.11).

The equation for the mean streamfunction for distribution (6.11) is easily obtained. It is convenient to change variables from the Fourier amplitudes used above to the configuration space relative vorticity $\zeta(\boldsymbol{r})$ as the independent variable defining the incompressible phase space. This is valid as can be seen by noting that the Jacobian of the transformation between the sets of variables is a constant. For normalizability, $P(\zeta)$ must vanish sufficiently rapidly in the limit of large $|\zeta|$; an equation for the mean streamfunction can be generated easily by using this property of P. We begin by calculating the integral of the total derivative of P over all phase space:

$$\begin{aligned}0 = \left\langle\frac{\delta}{\delta\zeta(\boldsymbol{r})}\right\rangle &\equiv \int \frac{\delta}{\delta\zeta(\boldsymbol{r})} P \prod_{\boldsymbol{r}'} \mathrm{d}\zeta(\boldsymbol{r}') \\ &= \int -a\left(\frac{\delta E}{\delta\zeta(\boldsymbol{r})}+\frac{\delta Q_G}{\delta\zeta(\boldsymbol{r})}\right) P \prod_{\boldsymbol{r}'} \mathrm{d}\zeta(\boldsymbol{r}'). \end{aligned} \tag{6.13}$$

Therefore,

$$0 = \left\langle\frac{\delta E}{\delta\zeta(\boldsymbol{r})}+\frac{\delta Q_G}{\delta\zeta(\boldsymbol{r})}\right\rangle. \tag{6.14}$$

Thus we need the variation of E, which can be calculated as follows

$$\begin{aligned}\frac{\delta E}{\delta\zeta(\boldsymbol{r})} &= \frac{\delta}{\delta\zeta(\boldsymbol{r})}\frac{1}{2}\iint (\nabla\psi)^2\,\mathrm{d}^2r' = \iint \nabla\psi\cdot\frac{\delta\nabla\psi}{\delta\zeta(\boldsymbol{r})}\,\mathrm{d}^2r' \\ &= -\iint \psi(\boldsymbol{r}')\frac{\delta\nabla^2\psi(\boldsymbol{r}')}{\delta\zeta(\boldsymbol{r})}\,\mathrm{d}^2r' = -\iint \psi(\boldsymbol{r}')\,\delta^2(\boldsymbol{r}-\boldsymbol{r}')\,\mathrm{d}^2r' \\ &= -\psi(\boldsymbol{r})\end{aligned} \tag{6.15}$$

as well as the variational derivative of Q_G which is calculated according to

$$\begin{aligned}\frac{\delta Q_G}{\delta\zeta(\boldsymbol{r})} &= \frac{\delta}{\delta\zeta(\boldsymbol{r})}\iint G(q(\boldsymbol{r}'))\,\mathrm{d}^2r' = \iint G'(q)\frac{\delta q(\boldsymbol{r}')}{\delta\zeta(\boldsymbol{r})} \\ &= -\iint G'(q)\,\delta^2(\boldsymbol{r}-\boldsymbol{r}')\,\mathrm{d}^2r' \\ &= G'(q).\end{aligned} \tag{6.16}$$

Combining these calculations we have

$$\langle\psi\rangle = \langle G'(q)\rangle, \tag{6.17}$$

which is the average of the relation (6.1′). This correspondence between ψ^{s} and the average $\langle\psi^{\mathrm{s}}\rangle$ is not an equivalence because one cannot equate $\langle G'(q)\rangle$ with $G'(\langle q\rangle)$. Equation (6.17) presents a closure problem – the average state in general can only be calculated by solving an infinite hierarchy of moment equations in a sequence of ascending degree, or approximately by some sort of closure scheme. It may be that for infinite resolution all energy is in the mean state as in the case with F' linear, and averages of products could then be replaced by products of averages yielding an equivalence in the infinite resolution limit. This is plausible because as noted above the solution q^{s} is a unique solution independent of the details of the particular choice of G.

There is an inconsistency involved in considering the distribution (6.11) for finite resolution that we have so far not mentioned. The problem is that only linear and quadratic vorticity invariants and energy can be expected to survive as dynamical invariants with the ultraviolet spectral truncation which is inherent in finite resolution simulation (cf. Seyler *et al.* 1975; Hald 1976). Moreover, the solutions of $\psi = F'(q)$ for other than linear F' are not even stationary in typical finite resolution models. Thus technically the only correct choice for the canonical statistics of the finite resolution dynamics is the energy–enstrophy ensemble. It would be incorrect, however, to try to deduce the properties of the continuum from a limit based only on the energy–enstrophy ensemble since the effects of the other invariants are then not recoverable. We have been proceeding rather from a hypothetical construct which maintains the relevant invariants at any resolution so as not to preclude valid continuum states. On the other hand, it is also interesting to consider simulations at high resolution and enquire about the conditions under which the effects of the conservation of the higher-order invariants might be significant for long (but finite) times.

The distribution (6.11) was designed to provide a normalizable statistical distribution for the consideration of perturbations from a stable state which has $F''(q^s)$ positive. The condition of normalizability alone however is not strong enough to indicate what the appropriate generalization of canonical equilibrium should be. Consider the distribution

$$P \propto \exp\{-a(E+Q_F)\}. \tag{6.18}$$

for arbitrary F. The first question, of course, is whether it is normalizable; for the present we assume that it is. The assumption of normalizability alone produces some interesting results. The calculation (6.13) would then produce $\langle\psi\rangle = \langle F'(q)\rangle$, assuming these averages exist. In addition normalizability also implies the definiteness of the operator $\{-\nabla^2 + F''\}$ in ensemble average; the calculation of the second variational derivative of P produces this result (see Appendix C). In the limit of infinite resolution we might hope that all the statistical weight would fall into the stationary states of (6.1). Again if there were only one solution for the given F we would expect the statistical mean to become equivalent to that unique stationary state. Of course, if there are multiple stable stationary solutions such an equivalence could not obtain since the averages would be a weighted sum over them all. More serious is the fact that the prescription (6.18) may have local minima at stable stationary points. It follows from the first functional variation of $\{E+Q_F\}$ that each stationary state is either an extremum or saddle of P. If the second variation is of definite sign then we have stability (at least in the linear sense); however, if that sign is positive for one solution and negative for another, then one or the other must be a state of minimal probability (locally in function space) in the distribution (6.18), which is inconsistent. These questions of normalizability and the proper extremal properties of the distribution for stable stationary states can be circumvented by using the microcanonical distribution

$$\delta(E-E_0)\,\delta(Q_F-Q_{F_0}),$$

where the subscript 0 indicates assigned or initial values. This distribution is always normalizable. In the limit of a large number of degrees of freedom, it may be possible to asymptotically obtain the appropriate generalized canonical equilibrium from the microcanonical (cf. Salmon *et al.* 1976 Appendix A). However, there is still the ambiguity of which invariant Q_F to use, and then why should not more than one be included.

7. Discussion

We have examined some properties of the inviscid equilibrium statistics of flow over topography. A general equivalence between nonlinear stability theory and inviscid statistical theory in the infinite resolution limit is strongly suggested. In the specific case of the minimum enstrophy state, we have demonstrated such an equivalence for barotropic flow over topography (including the case driven by dynamically interacting large-scale flow). We have discussed some of the possibilities as well as difficulties involved in demonstrating this equivalence in its most general form. These considerations raise many questions about the formulation of inviscid statistical mechanics. We shall conclude by noting some further interesting questions.

It should be clear that in the calculation of the 'infinite resolution limit' of the energy–enstrophy ensemble, we have assumed a limiting process in which the time is first allowed to go to infinity for a finite resolution. Then, does our result have anything to do with the long-time behaviour of the infinite resolution flow? In two-dimensions with analytic initial conditions there are no finite time singularities, which implies that the conservation laws apply for all time (cf. Rose & Sulem 1978). Consider a flow which initially has an energy spectrum with an ultraviolet cutoff. In a turbulence scenario, scrambling of phase information persists for all time, and hence, with the conservation of energy and enstrophy, the mechanism for a tendency toward canonical equilibrium operates at all times no matter what the resolution. The turbulent flow develops with the excitation of higher and higher scales, and a self-similar cascade is established at high wavenumbers. The spectrum never has the shape of a finite resolution equilibrium spectrum; however, in such a self-similar cascade, the eddy energy at all but the largest scale tends to vanish as time approaches infinity. Thus a turbulent self-similar evolution would actually tend toward the infinite resolution limit of canonical equilibrium. However, it is possible that as the spectral energy density decreases, phase-locking will result in an asymptotic tendency toward a coherent structure that would not participate in the turbulent cascade and hence violate the statistical prediction. Whether the statistical theory can in fact be generalized to include this latter possibility is an intriguing question.

Further questions are raised by a reappraisal of the arguments in support of the energy–enstrophy ensemble. One such argument is based on the phase space structure of the surfaces representing the higher-order vorticity invariants. The integral of any power of the potential vorticity is an invariant. In a spectral representation these invariants take the form:

$$\iint \mathrm{d}^2 r q^n = \Sigma\, q_{k_1} q_{k_2} \cdots q_{k_n} \delta_{k_1+\ldots+k_n,\,0},$$

where the sum is over all n wavevectors. It seems plausible that, in contrast to the energy and enstrophy ellipsoids, the constant level surfaces of these invariants for $n > 2$ are very complicated. The presumption is that these complicated, interleaved hypersurfaces intersect the energy–enstrophy hypersurface in a way which samples *that* surface well, and so coarse-grained statistical averages are accurately obtained simply by averages over the intersection of the energy and enstrophy hypersurfaces. Then the argument is that the restriction to these higher-order invariant hypersurfaces does not significantly affect the statistics. Numerical simulation of inviscid flow evolving from randomly generated initial conditions supports this picture in that the evolution is toward the canonical equilibrium spectrum (Fox & Orszag 1973;

Basdevant & Sadourny 1975; Seyler *et al.* 1975; Kells & Orszag 1978; Carnevale 1982). On the other hand the possible existence of nonlinearly stable flows is at variance with this picture of 'mixing' on the energy–enstrophy hypersurface. With reasonably high resolution one might find non-mixing behaviour by choosing initial conditions near a stable equilibrium. An example which has motivated some of our considerations is the modon. A nonlinear stability theorem for this strongly nonlinear, dipolar, coherent structure remains elusive; however, numerical simulations indicate that it has a remarkable degree of stability (McWilliams *et al.* 1981). Similarly on the f-plane we must consider the behaviour of stable minimum enstrophy vortices (Leith 1984). Furthermore, Malanotte Rizzoli (1982) demonstrates that certain solitary waves propagating over rough topography persist indefinitely in long-term numerical simulations. Such resistance to relaxation to canonical equilibrium suggests that certain regions of the energy–enstrophy hypersurface do not effectively mix with the whole. Thus even though 'most' ensembles may be expected to relax toward the energy–enstrophy ensemble, there is a set for which an ensemble based on higher-order invariants seems to be required for a proper statistical treatment.

Finally we note the work of Thompson (1974), who shows that the assumptions of spatially local dynamics and total vorticity, energy and enstrophy conservation imply the higher-order conservation laws. Clearly the assumption of spatial locality is very strong, but how it leads to the Euler equation from the low-order invariants is quite remarkable. The implications of this result for statistical mechanics are not clear to us; however, we mention it here because it may eventually help elucidate the proper specification of statistics for flows near stable equilibria as discussed in §6.

We hope to pursue the questions raised in this section in future work.

We especially wish to thank Rick Salmon and Geoff Vallis for many helpful discussions of this work. We also wish to thank Henry Abarbanel, Roberto Benzi, Annalisa Griffa, Frank Henyey, Darryl Holm, Roberto Purini, and William Young for their help at various stages in the development of the ideas presented here. One of us (G.F.C.) was supported by the Office of Naval Research (USN N00014-85-C-0104) during this work.

Appendix A. Two-scale derivation of the form-drag equation

Here we shall use the techniques of multiple scale analysis to obtain the equation of evolution of the large-scale velocity U which was derived above simply from energy conservation. This calculation expands on that given by Hart (1979).

Anticipating the effect of the large-scale variation of the Coriolis parameter, we introduce a large-scale variable in the y-direction. Thus we take

$$y_1 = y, \quad y_2 = \epsilon y,$$

where ϵ is a small parameter expressing the ratio of small scales tolarge (i.e. $\epsilon = l/L$ and y_2 represents variation on the large scale). We then introduce the ordered decomposition

$$h(x,y) = \frac{1}{\epsilon} h_0(y_2) + h_1(x, y_1, y_2) + \epsilon h_2(x, y_1, y_2) + O(\epsilon^2), \tag{A 1}$$

where we include the large-scale variation of the Coriolis parameter in the h_0 term (i.e. $h_0 = \beta y_2$ so that $(1/\epsilon)h_0 = \beta y$). Also the same decomposition will be found adequate for ψ:

$$\psi(x,y) = \frac{1}{\epsilon}\psi_0(y_2) + \psi_1(x, y_1, y_2) + \epsilon\psi_2(x, y_1, y_2) + O(\epsilon^2). \tag{A 2}$$

Next we substitute these representations into the evolution equation

$$\nabla^2\dot{\psi} + J(\psi, \nabla^2\psi + h) = 0.$$

For a compact notation we introduce

$$\partial_i = \frac{\partial}{\partial y_i},$$

and

$$\nabla_1 = (\partial_x, \partial_1),$$

with $i = 1, 2$. Thus we have the Laplacian decomposed according to

$$\nabla^2 = \nabla_1^2 + \epsilon 2\partial_1\partial_2 + \epsilon^2\partial_2^2. \tag{A 3}$$

Substituting these decompositions into the evolution equation we obtain an ordered series of equations. The lowest non-trivial order gives

$$\nabla_1^2\dot{\psi}_1 = -J_1(\psi_1, q_1) + (\partial_2\psi_0)\partial_x q_1 - (\partial_x\psi_1)(\partial_2 h_0), \tag{A 4}$$

where

$$q_1 \equiv \nabla_1^2\psi_1 + h_1,$$

and

$$J_i(A, B) \equiv \partial_x A\, \partial_i B - \partial_x B\, \partial_i A,$$

with $i = 1, 2$. If we now take the limit

$$\psi_0 \to -Uy_2,$$

and

$$h_0 \to \beta y_2,$$

this yields

$$\nabla_1^2\dot{\psi}_1 = -J_1(\psi_1, q_1) - U\partial_x q_1 - \beta(\partial_x\psi_1),$$

or equivalently

$$\nabla_1^2\dot{\psi}_1 = -J_1(\psi_1 - Uy,\ \nabla_1^2\psi_1 + h_1 + \beta y), \tag{A 5}$$

the desired result.

The next order in ϵ yields

$$\left.\begin{aligned}
&\partial_2^2\dot{\psi}_0 + 2\partial_1\partial_2\dot{\psi}_1 + \nabla_1^2\dot{\psi}_2 \\
&\quad = -J_1(\psi_2, q_1) - J_1(\psi_1,\ 2\partial_1\partial_2\psi_1 + \nabla_1^2\psi_2 + h_2) \\
&\qquad + (\partial_2\psi_0)\partial_x(2\partial_1\partial_2\psi_1 + \nabla_1^2\psi_2 + h_2) - (\partial_x\psi_2)(\partial_2 h_0) - J_2(\psi_1, q_1). \\
&\quad = -J_1(\psi_2, q_1) - J_1(\psi_1,\ 2\partial_1\partial_2\psi_1 + \nabla_1^2\psi_2 + h_2) \\
&\qquad + \partial_x\{(\partial_2\psi_0)(2\partial_1\partial_2\psi_1 + \nabla_1^2\psi_2 + h_2) - \psi_2\partial_2 h_0\} - J_2(\psi_1, q_1).
\end{aligned}\right\} \tag{A 6}$$

The J_1 terms are full divergences in x and y_1, and thus they do not contribute to the integral over the small-scale periodic domain, and similarly all the other terms in (A 6) which are derivatives in the periodic variables will not contribute. Integration over the periodic domain thus yields

$$\begin{aligned}
\partial_2^2\dot{\psi}_0 &= -\iint \mathrm{d}x\,\mathrm{d}y_1\, J_2(\psi_1, q_1) \\
&= -\partial_2 \iint \mathrm{d}x\,\mathrm{d}y_1\, h_1\,\partial_x\psi_1.
\end{aligned}$$

The last line is obtained after some manipulation and integration by parts. Now we integrate the y_2 variable once to obtain

$$\partial_2 \dot{\psi}_0 = -\iint \mathrm{d}x\,\mathrm{d}y_1\, h_1\, \partial_x \psi_1 + C, \tag{A 7}$$

where in the unforced problem we take the integration constant C to be zero. Thus in the limit

$$\psi_0 \rightarrow -Uy_2,$$

which we can only take at this point, we have the desired result:

$$\dot{U} = \iint \mathrm{d}x\,\mathrm{d}y_1\, h_1 \frac{\partial \psi_1}{\partial x}. \tag{A 8}$$

It should be possible to derive the modified energy and potential enstrophy conservation laws directly by writing these quantities for the fields in this two-scale representation. The energy per unit area is given by

$$\begin{aligned} E &= \frac{1}{2}\frac{1}{L^2}\int_0^L\int_0^L \mathrm{d}x\,\mathrm{d}y(\nabla\psi)^2, \\ &= \frac{1}{2}\frac{1}{L^2}\int_0^L\int_0^L \mathrm{d}x\,\mathrm{d}y\left[(\nabla_1\psi_1)^2 + \frac{\partial\psi_0}{\partial y_2}\frac{\partial\psi_1}{\partial y_1} + \left(\frac{\partial\psi_0}{\partial y_2}\right)^2 + O(\epsilon^2)\right]. \end{aligned}$$

We can write the integral over all space as a sum of integrals over the small-scale domains. By assumption, in the integrand of any integral over an area $l \times l$ the dependence on the y_2 variable is weak. The total integral is evaluated as a sum of L^2/l^2 such small-scale integrals. Assuming periodicity on the small scale then yields

$$E = \tfrac{1}{2}U^2 + \frac{1}{2}\frac{1}{l^2}\int_0^l\int_0^l \mathrm{d}x\,\mathrm{d}y(\nabla_1\psi_1)^2. \tag{A 9}$$

For the potential enstrophy per unit area we have

$$\begin{aligned} Q &= \frac{1}{2}\frac{1}{L^2}\int_0^L\int_0^L \mathrm{d}x\,\mathrm{d}y\, q^2, \\ &= C + \frac{1}{L^2}\int_0^L\int_0^L \mathrm{d}x\,\mathrm{d}y\left(\frac{1}{\epsilon}h_0 q_1 + \tfrac{1}{2}q_1^2 + h_0\partial_2^2\psi_0 + \partial_1\partial_2\psi_1 + O(\epsilon)\right). \end{aligned}$$

Here C is just a constant coming from the static integrals involving only h. For the term $h_0\partial_2^2\psi_0$ we are forced into a somewhat *ad hoc* treatment. To obtain the form we desire requires an integration by parts over the whole space; although we take the local limiting form to be $\psi_0 \rightarrow -Uy_2$, we must assume boundary terms bring ψ to 0 so we can eliminate the boundary integral. The other terms are handled as in the energy calculation above. Thus

$$Q = \beta U + \frac{1}{2}\frac{1}{l^2}\int_0^l\int_0^l \mathrm{d}x\,\mathrm{d}y\, q^2 + C + O(\epsilon). \tag{A 10}$$

Appendix B. Conservation laws of the linearized dynamics

Consider the evolution of the flow

$$\frac{\partial q}{\partial t} + J(\psi, q) = 0,$$

linearized about the stationary state

$$\psi^{\mathrm{s}} = F'(q^{\mathrm{s}}).$$

Substituting $\psi = \psi^s + \delta\psi$ into the evolution equation yields

$$\begin{aligned}\frac{\partial \delta q}{\partial t} &= -J(\psi^s + \delta\psi, q + \delta q)\\ &= -J(F(q^s) + \delta\psi, q^s + \delta q)\\ &= -J(\delta\psi, q^s) - F''(q^s)\, J(q^s, \delta q) - J(\delta\psi, \delta q)\\ &= -J(q^s, -\delta\psi + F''(q^s)\,\delta q) - J(\delta\psi, \delta q)\\ &= (\delta\dot{q})_{\text{lin}} - J(\delta\psi, \delta q),\end{aligned}$$

where $\delta q \equiv \nabla^2 \delta\psi$, and the linearized dynamics are determined by

$$(\delta\dot{q})_{\text{lin}} \equiv -J(q^s, -\delta\psi + F''(q^s)\,\delta q). \tag{B 1}$$

The invariance of the quadratic integral,

$$\iint \delta q(-\nabla^{-2} + F''(q^s))\,\delta q\, \mathrm{d}^2 r, \tag{B 2}$$

in the linear dynamics can be tested by direct calculation with (B 1). Alternatively, more insight is gained by examining the nonlinear conservation laws order by order in the perturbation. The energy conservation law is

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}t} &= -\iint \mathrm{d}^2 r \psi \nabla^2 \dot{\psi},\\ &= -\iint \mathrm{d}^2 r(\psi^s + \delta\psi)\,((\delta\dot{q})_{\text{lin}} - J(\delta\psi, \delta q)),\\ &= -\iint \mathrm{d}^2 r[\psi^s(\delta\dot{q})_{\text{lin}}] + [\delta\psi(\delta\dot{q})_{\text{lin}} - \psi^s J(\delta\psi, \delta q)] - [\delta\psi J(\delta\psi, \delta q)].\end{aligned} \tag{B 3}$$

Since the perturbation may be arbitrary, each of these brackets in the last line representing different orders in the perturbation must vanish independently. The first term shows that $\iint \psi^s \delta q$ is conserved by the linear dynamics. Now consider the vorticity invariants. The nonlinear dynamics conserves

$$\Gamma = \iint \mathrm{d}^2 r \gamma(q),$$

where γ is an arbitrary function. The conservation of Γ can also be written order by order in the perturbation:

$$\begin{aligned}\frac{\mathrm{d}\Gamma}{\mathrm{d}t} &= \iint \mathrm{d}^2 r \gamma'(q)\,\dot{q},\\ &= \iint \mathrm{d}^2 r[\gamma'(q^s) + \gamma''(q^s)\,\delta q + O(\delta q^2)]\,[(\delta\dot{q})_{\text{lin}} - J(\delta\psi, \delta q)],\\ &= \iint \mathrm{d}^2 r[\gamma(q^s)\,(\delta\dot{q})_{\text{lin}}] + [\gamma''(q^s)\,\delta q(\delta\dot{q})_{\text{lin}} - \gamma'(q^s)\, J(\delta\psi, \delta q)] + O(\delta q^3).\end{aligned} \tag{B 4}$$

Again there must be an order by order cancellation, so that we immediately have that

$$\iint \gamma(q^s)\,\delta q,$$

is conserved by the linear dynamics; note that γ is completely arbitrary and need have no relation to ψ^s. Although these rate equations for E and Γ do not at second order independently yield further linearly conserved quantities, the addition of their second-order terms with the identification $\gamma = F$ is such that the nonlinear part of the dynamics cancels identically. Thus we have that (B 2) is conserved by the linear dynamics.

Appendix C. Calculation of $\langle -\nabla^2 + F'' \rangle$

We assume the distribution

$$P \propto \exp\{-a(E+Q_F)\} \tag{C 1}$$

is normalizable. The result is that on average $a\{-\nabla^2 + F''\}$ is positive. The calculation of the second variational derivative of P produces this result; that is, we consider

$$\begin{aligned}
\left\langle \frac{\delta}{\delta\zeta(\boldsymbol{r})} \frac{\delta}{\delta\zeta(\boldsymbol{r}')} \right\rangle &\equiv \int \frac{\delta}{\delta\zeta(\boldsymbol{r})} \frac{\delta}{\delta\zeta(\boldsymbol{r}')} P \prod_{\boldsymbol{r}''} \mathrm{d}\zeta(\boldsymbol{r}'') \\
&= -a \int \frac{\delta}{\delta\zeta(\boldsymbol{r})} [(-\psi(\boldsymbol{r}') + F'(q(\boldsymbol{r}')))P] \prod_{\boldsymbol{r}''} \mathrm{d}\zeta(\boldsymbol{r}'') \\
&= a^2 \int [-\psi(\boldsymbol{r}') + F'(q(\boldsymbol{r}'))][-\psi(\boldsymbol{r}) + F'(q(\boldsymbol{r}))] P \prod_{\boldsymbol{r}''} \mathrm{d}\zeta(\boldsymbol{r}'') \\
&\quad - a \int \left[-\frac{\delta\psi(\boldsymbol{r}')}{\delta\zeta(\boldsymbol{r})} + F''(q(\boldsymbol{r}')) \frac{\delta q(\boldsymbol{r}')}{\delta\zeta(\boldsymbol{r})} \right] P \prod_{\boldsymbol{r}''} \mathrm{d}\zeta(\boldsymbol{r}'') \\
&= a^2 \langle [-\psi(\boldsymbol{r}') + F'(q(\boldsymbol{r}'))][-\psi(\boldsymbol{r}) + F'(q(\boldsymbol{r}))] \rangle \\
&\quad - a \left\langle -\frac{\delta\psi(\boldsymbol{r}')}{\delta\zeta(\boldsymbol{r})} + F''(q(\boldsymbol{r}'))\, \delta^2(\boldsymbol{r}-\boldsymbol{r}') \right\rangle .
\end{aligned} \tag{C 2}$$

The operator is to be understood in terms of its action on an arbitrary test function. Explicitly we have for arbitrary $\eta(\boldsymbol{r})$

$$\begin{aligned}
a \iint \mathrm{d}^2 r\, \mathrm{d}^2 r'\, \eta(\boldsymbol{r}) \left\langle -\frac{\delta\psi(\boldsymbol{r}')}{\delta\zeta(\boldsymbol{r})} + F''(q(\boldsymbol{r}'))\, \delta^2(\boldsymbol{r}-\boldsymbol{r}') \right\rangle \eta(\boldsymbol{r}') \\
= a^2 \iint \mathrm{d}^2 r\, \mathrm{d}^2 r'\, \eta(\boldsymbol{r}) \langle [-\psi(\boldsymbol{r}') + F'(q(\boldsymbol{r}'))][-\psi(\boldsymbol{r}) + F'(q(\boldsymbol{r}))] \rangle\, \eta(\boldsymbol{r}'),
\end{aligned} \tag{C 3}$$

or in a more illuminating form

$$a \int \mathrm{d}^2 r\, \eta(\boldsymbol{r}) \left[-\frac{1}{\nabla^2} + \langle F''(q) \rangle \right] \eta(\boldsymbol{r}) = a^2 \left\langle \left[\int \mathrm{d}^2 r\, \eta(\boldsymbol{r}) (-\psi(\boldsymbol{r}) + F'(q(\boldsymbol{r}))) \right]^2 \right\rangle . \tag{C 4}$$

Thus the left-hand side is indeed positive for arbitrary test function η. This identity is easily checked for the simple case with quadratic F by direct substitution of the canonical averages (3.7).

REFERENCES

ARNOL'D, V. I. 1965 On conditions for nonlinear stability of plane stationary curvilinear flows of an ideal fluid. *Sov. Math. Dokl.* **6**, 773–777.

ARNOL'D, V. I. 1969 On an a priori estimate in the theory of hydrodynamical stability. *Am. Math. Soc. Trans.* **79**, 267–269.

BASDEVANT, C. & SADOURNY, R. 1975 Ergodic properties of inviscid truncated models of two-dimensional incompressible flows. *J. Fluid Mech.* **69**, 673–688.

BRETHERTON, F. P. & HAIDVOGEL, D. B. 1976 Two-dimensional turbulence above topography. *J. Fluid Mech.* **78**, 129–154.

CARNEVALE, G. F., FRISCH, U. & SALMON, R. 1981 H-theorems in statistical fluid dynamics. *J. Phys.* A **14**, 1701–1718.

CARNEVALE, G. F. 1982 Statistical features of the evolution of two-dimensional turbulence. *J. Fluid Mech.* **122**, 143–153.

CHARNEY, J. G. & DEVORE, J. G. 1979 Multiple flow equilibria in the atmosphere and blocking. *J. Atmos. Sci.* **36**, 1205–1216.

FOFONOFF, N. P. 1954 Steady flow in a frictionless homogeneous ocean. *J. Mar. Res.* **13**, 254–262.

FOX, D. G. & ORSZAG, S. A. 1973 Inviscid dynamics of two-dimensional turbulence. *Phys. Fluids* **16**, 169–171.

FREDERIKSEN, J. S. 1982 Eastward and westward flows over topography in nonlinear and linear barotropic models. *J. Atmos. Sci.* **39**, 2477–2489.

FREDERIKSEN, J. S. & CARNEVALE, G. F. 1986 Stability properties of exact nonzonal Arnol'd solutions for flow over topography. *Geophys. Astrophys. Fluid Dynamics* **35**, 173–207.

HALD, O. H. 1976 Constants of motion in two-dimensional turbulence. *Phys. Fluids* **19**, 914–915.

HART, J. E. 1979 Barotropic quasigeostrophic flow over anisotropic mountains. *J. Atmos. Sci.* **36**, 1736–1746.

HENYEY, F. S. 1983 Hamiltonian description of stratified fluid dynamics. *Phys. Fluids* **26**, 40–47.

HOLM, D. D., MARSDEN, J. E., RATIU, J. E. & WEINSTEIN, A. 1985 Nonlinear stability of fluid and plasma equilibria. *Phys. Rep.* **123**, 1–116.

KATZ, A. 1967 *Principles of Statistical Mechanics.* Freeman.

KELLS, L. C. & ORSZAG, S. A. 1978 Randomness of low order models of two-dimensional inviscid dynamics. **21**, 162–168.

KRAICHNAN, R. H. 1975 Statistical dynamics of two-dimensional flow. *J. Fluid Mech.* **67**, 155–175.

KRAICHNAN, R. H. & MONTGOMERY, D. 1980 Two-dimensional turbulence. *Rep. Prog. Phys.* **43**, 547–618.

LEITH, C. E. 1984 Minimum enstrophy vortices. *Phys. Fluids* **27**, 1388–1395.

MALANOTTE RIZZOLI, P. 1982 The stability of planetary solitary waves. In *Proceedings of the International School of Physics, Enrico Fermi* (ed. A. R. Osborne & P. Malanotte Rizzoli), pp. 148–183. North-Holland.

MCWILLIAMS, J. C., FLIERL, G. R., LARICHEV, V. D. & REZNIK, G. M. 1981 Numerical studies of barotropic modons. *Dyn. Atmos. Oceans* **5**, 219–238.

MONTGOMERY, D. 1976 A BBGKY framework for fluid turbulence. *Phys. Fluids* **19**, 802–810.

MONTGOMERY, D. 1985 Maximal entropy in fluid and plasma turbulence: a review. In *Maximum-Entropy and Baysian Methods in Inverse Problems* (ed. C. R. Smith & W. T. Grandy), Reidel.

ORSZAG, S. A. 1970 Analytical theories of turbulence. *J. Fluid Mech.* **41**, 363–386.

PEDLOSKY, J. 1979 *Geophysical Fluid Dynamics.* 624 pp. Springer.

PETRONI, R., PIERINI, S. & VULPIANI, A. 1986 About the double cascade mechanism and instabilities of steady solutions in two-dimensional flow. (preprint).

PURINI, R. & SALUSTI, E. 1984 Non-linear hydrodynamic stability of ocean flows. **30**, 261–270.

ROSE, H. A. & SULEM, P. L. 1978 Fully developed turbulence and statistical mechanics. *J. Physique* **39**, 441–484.

SALMON, R., HOLLOWAY, G. & HENDERSHOTT, M. C. 1976 The equilibrium statistical mechanics of simple quasi-geostrophic models. *J. Fluid Mech.* **75**, 691–703.

SALMON, R. 1982 Hamilton's principle and Ertel's theorem. In *Mathematical Methods in Hydrodynamics and Integrability in Dynamical Systems* (ed. M. Tabor & Y. M. Treve), *Am. Inst. Phys. Proc.* **88**, 127–135.

SEYLER, C. E., YEHUDA, S., MONTGOMERY, D. & KNORR, G. 1975 Two-dimensional turbulence in inviscid fluids or guiding center plasmas. *Phys. Fluids* **18**, 803–813.

THOMPSON, P. D. 1974 The integral invariants of two-dimensional flow and their role in the theory of two-dimensional turbulence. *J. Atmos. Sci.* **31**, 1453–1457.

VALLIS, G. K. 1985 Instability and flow over topography. *Geophys. Astrophys. Fluid Dyn.* **34**, 1–38.